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ASYMPTOTIC EXPANSION OF SOLUTION OF BVP WITH INITIAL JUMPS FOR SINGULARLY PERTURBED INTEGRO-DIFFERENTIAL EQUATION

Abstract: we consider the undivided boundary value problem for singularly perturbed linear integro-differential equations of $n-$th order with integral Fredholm terms, which has the phenomenon m_1-th order of an initial jump of order at the left end of the segment under consideration. The regular and boundary layer parts of the asymptotic expansion of solutions are determined. The regular terms of the asymptotics are constructed in the form of integral-differential equations, which differ from the usual unperturbed integro-differential equations by the presence of additional terms, called the initial jumps of the integral terms. The magnitudes of these initial jumps are determined. The boundary conditions for the regular terms of the asymptotics also contain additional terms, called the initial jumps of the derivatives of the m_1-th order. Thus, to determine the regular terms of the asymptotics, we obtain the boundary conditions for linear integro-differential equations with an additional parameter. To determine the boundary layer terms of the asymptotic expansion of solutions, initial problems are obtained for homogeneous and inhomogeneous ordinary differential equations with constant coefficients. Exponential estimates are obtained for the boundary layer terms of the asymptotics. The theorem of existence, uniqueness and both of the symmetric representation of solutions with an estimate of the remainder of the asymptotics is formulated. It is established that the constructed asymptotic approximation to the solution of the original singularly perturbed integro-differential boundary value problem is uniform on the entire considered interval.

Key words: singular perturbation, small parameter, asymptotic expansion, the initial jump, the boundary layer.

Introduction. Mathematics studies natural processes using mathematical models for them. Any mathematical model is somewhat approximate. When constructing the mathematical models, one tries to capture all essential, dominant feature of the process. On the other hand, the model should be “simple” enough to allow the analytical and numerical treatment leading to the information one wants to obtain about the process. A variety of models in physics, chemical kinetics, mathematical biology, and many other fields are quite naturally formulated in terms of differential equations. During the derivation of the model equations, some terms whose influence on the process is supposed to be negligible are often not taken into account. As a result, the model might be simplified considerably. Such simplifications often rely on physical intuition. Note that other mathematical school of singularly perturbed equations in Kazakhstan and abroad investigate only boundary value problems, which do not have an initial jump. We say a function $y(x,\varepsilon)$ is a singular perturbation of $y(x,0)$ if $y(x,0)$ fails to approximate $y(x,\varepsilon)$ for all x of interest when ε is small. Uniformly valid approximations for such functions can often be found by the modified boundary function method. In our previous works in [1-13] we considered the initial and boundary value problems with the initial jump for differential and integro-differential equations in the stable case. In the article [14] the constructive formula for solutions of boundary value problems for singularly perturbed higher order linear integro-differential equations has been obtained. The asymptotic properties of solutions which are similar to the properties of solutions of differential equations are shown. The difference was that the equations for the coefficients of the asymptotic expansion in the case of integro –differential equations are some what more complicated than in the case of differential equations. Thus, we can say that the addition of integral terms to the equation leads to some complication of the algorithm for constructing the asymptotic expansion of the solution and the qualitative behavior of the solution will not change. Therefore the problem arises in identifying such asymptotic properties of solutions of integro-differential equations that are not characteristic of differential equations, investigating the problems of this type when the present integral terms will lead to a qualitative change in the asymptotic behavior of solutions of differential equations.

Scientific novelty. It is shown that solutions of singularly perturbed general boundary value problems tend to solutions of the corresponding modified degenerate problems. The modified degenerate problems have been constructed. The values of the initial jumps of the solutions and of the integral terms have been found. Asymptotic expansions of solutions of BVP with ordered boundary conditions and general boundary value problems with any degree of accuracy with respect to a small parameter have been constructed.

Theoretical and practical significance of the research. The results obtained an important contribution to the development of the theory of singularly perturbed integro-differential equations. Taking into account that the equation is one of the most important classes of singularly perturbed equations, they are mathematical models that study a wide variety of processes in physics, chemistry, biology, engineering, and others.

Materials and methods.

Consider the following singularly perturbed integro-differential equation

$$L_\varepsilon y \equiv \varepsilon y^{(n)} + A_1(t)y^{(n-1)} + \ldots + A_n(t)y = F(t) + \int_0^1 \sum_{i=0}^{m_1+1} H_i(t,x)y^{(i)}(x,\varepsilon)dx \qquad (1)$$

with nonlocal boundary condition

$$h_i y(t,\varepsilon) \equiv \sum_{j=0}^{m_i} \alpha_{ij} y^{(j)}(0,\varepsilon) + \sum_{j=0}^{l_i} \beta_{ij} y^{(j)}(1,\varepsilon) = a_i, \quad i = \overline{1,n}, \qquad (2)$$

where $\varepsilon > 0$ is small parameter, $\alpha_{ij}, \beta_{ij}, a_i \in R$ are known constants independent of ε and $\alpha_{1,m_1} \neq 0, n-1 > m_1 > m_2 \geq \ldots \geq m_n,\ n-1 > l_i, i = \overline{1,n}$.

Assume that the following conditions hold:

(C1) $A_i(t)$, $F(t)$, $i = \overline{1,n}$ are sufficiently smooth functions on the interval $[0,1]$;

(C2) $A_1(t) \geq \gamma = const > 0, 0 \leq t \leq 1$;

(C3) Functions $H_i(t,x), i = 0,\ldots,m_1+1$ are defined and sufficiently smooth in the domain $D = (0 \leq t \leq 1,\ 0 \leq x \leq 1)$;

(C4) $\overline{\Delta} = \begin{vmatrix} h_2 y_{10}(t) & \ldots & h_2 y_{n-1,0}(t) \\ \ldots & \ldots & \ldots \\ h_n y_{10}(t) & \ldots & h_n y_{n-1,0}(t) \end{vmatrix} \neq 0$, where $y_{i0}(t), i = 1,\ldots,n-1$ is the fundamental set of solutions of the following homogeneous differential equation

$$L_0 y(t) \equiv A_1(t) y^{(n-1)}(t) + \ldots + A_n(t) y\ (t) = 0.$$

(C5) $\lambda = 1$ is not an eigenvalue of the kernel $H(t,s,\varepsilon)$.

(C6) $$\bar{\varpi} = \begin{vmatrix} 1+\bar{d}_{11} & \bar{d}_{12} & \dots & \bar{d}_{1n} \\ \bar{d}_{21} & 1+\bar{d}_{22} & \dots & \bar{d}_{2n} \\ \dots & \dots & \dots & \dots \\ \bar{d}_{n1} & \bar{d}_{n2} & \dots & 1+\bar{d}_{nn} \end{vmatrix} \neq 0.$$

(C7) Number 1 is not an eigenvalue of the kernel $\overline{H}(t,s)$.

For the solution of the problem (1),(2) are valid the following limiting equalities [5]:

$$\lim_{\varepsilon\to 0} y^{(j)}(t,\varepsilon) = \bar{y}^{(j)}(t),\ j=\overline{0,m_1-1},\ 0\le t\le 1,$$
$$\lim_{\varepsilon\to 0} y^{(m_1+j)}(t,\varepsilon) = \bar{y}^{(m_1+j)}(t),\ j=\overline{0,n-1-m_1},\ 0< t\le 1, \tag{3}$$

where $\bar{y}(t)$ is the solution of the degenerate problem, Δ_0 is the initial jump of the solution,

$$L_0\bar{y} \equiv A_1(t)\bar{y}^{(n-1)}(t) + \sum_{i=2}^{n} A_i(t)\bar{y}^{(n-i)}(t) = F(t) + \int_0^1 \sum_{i=0}^{m_1+1} H_i(t,x)\bar{y}^{(i)}(x)dx + \Delta(t) \tag{4}$$
$$h_1\bar{y}(t) \equiv \sum_{j=0}^{m_1} \alpha_{1j}\bar{y}^{(j)}(0) + \sum_{j=0}^{l_1} \beta_{1j}\bar{y}^{(j)}(1) = a_1 - \alpha_{1,m_1}\Delta_0,$$
$$h_i\bar{y}(t) \equiv \sum_{j=0}^{m_i} \alpha_{ij}\bar{y}^{(j)}(0) + \sum_{j=0}^{l_i} \beta_{ij}\bar{y}^{(j)}(1) = a_i, \quad i=\overline{2,n}$$

From (3) it follows that the solution $y(t,\varepsilon)$ of the general boundary value problem (1) and (2) converges to the solution $\bar{y}(t)$ of the modified degenerate problem (4) as $\varepsilon\to 0$. We note that the limits for $y^{(m_1+j)}(t,\varepsilon),\ j=\overline{0,n-1-m_1}$ are not uniform on the interval $0\le t\le 1$. They are uniform on the interval $0<t_0\le t\le 1$, where t_0 is sufficiently small but fixed number as $\varepsilon\to 0$. In the work will be constructed uniformly asymptotic expansion of the solution of the problem (1), (2) on the interval $0\le t\le 1$. Therefore near $t=0$ the zone appears in which no matter how small ε it is, the solution of the original perturbed equation is strongly different from the solution of the degenerate one. This zone is called the boundary layer zone.

Since the solution of the problem (1) and (2) has the m_1-th order initial jump at the point $t=0$, we seek the asymptotic expansion of the solution of the problem (1), (2) in the next form:

$$y(t,\varepsilon) = y_\varepsilon(t) + \varepsilon^{m_1} w_\varepsilon(\tau),\quad \tau=\frac{t}{\varepsilon}, \tag{5}$$

where $y_\varepsilon(t)$ is a regular part of the asymptotic and $w_\varepsilon(\tau)$ is a boundary layer part, those can be represented in the form:

$$y_\varepsilon(t) = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \dots$$
$$w_\varepsilon(\tau) = w_0(\tau) + \varepsilon w_1(\tau) + \varepsilon^2 w_2(\tau) + \dots \tag{6}$$

Substituting the series (5) into (1), we obtain the following equalities:

$$\varepsilon\left[y_\varepsilon^{(n)}(t) + \frac{\varepsilon^{m_1}}{\varepsilon^n} w_\varepsilon^{(n)}(\tau)\right] + A_1(t)\left[y_\varepsilon^{(n-1)}(t) + \frac{\varepsilon^{m_1}}{\varepsilon^{n-1}} w_\varepsilon^{(n-1)}(\tau)\right] + \dots +$$

$$+A_n(t)\left[y_\varepsilon(t)+\varepsilon^{m_1}w_\varepsilon(\tau)\right]=F(t)+\int_0^1\left\{H_0(t,x)\left[y_\varepsilon(x)+\varepsilon^{m_1}w_\varepsilon\left(\frac{x}{\varepsilon}\right)\right]+\ldots+\right.$$

$$\left.+H_{m_1}(t,x)\left[y_\varepsilon^{(m_1)}(x)+\overset{(m_1)}{w_\varepsilon}\left(\frac{x}{\varepsilon}\right)\right]+H_{m_1+1}(t,x)\left[y_\varepsilon^{(m_1+1)}(x)+\frac{1}{\varepsilon}\overset{(m_1+1)}{w_\varepsilon}\left(\frac{x}{\varepsilon}\right)\right]\right\}dx \tag{7}$$

By replacing the integral expression $s=\dfrac{x}{\varepsilon}$ on the right-hand side of the equation (7), we get the improper integral

$$J(t,\varepsilon)=\int_0^\infty\Big[\varepsilon^{m_1+1}H_0(t,\varepsilon s)w_\varepsilon(s)+\varepsilon^{m_1}H_1(t,\varepsilon s)(s)\dot{w}(s)+\ldots+\varepsilon H_{m_1}(t,\varepsilon s)\overset{(m_1)}{w_\varepsilon}(s)+$$

$$+H_{m_1+1}(t,\varepsilon s)(s)\overset{(m_1+1)}{w_\varepsilon}(s)\Big]ds \tag{8}$$

The improper integral in (8) converges and the second sum in (8) is vanished, because $O\left(\exp\left(-\gamma\dfrac{t}{\varepsilon}\right)\right)$ is less than any power of ε, as $\varepsilon\to 0$. We write separately the coefficients depending on t and on τ we obtain the following equalities for $y_\varepsilon(t)$ and $w_\varepsilon(\tau)$:

$$\varepsilon y_\varepsilon^{(n)}(t)+\sum_{i=1}^{n}A_i(t)y_\varepsilon^{(n-i)}(t)=F(t)+\int_0^1\sum_{i=0}^{m_1+1}H_i(t,x)y_\varepsilon^{(i)}(x)dx+$$

$$+\int_0^\infty\sum_{i=0}^{m_1+1}\varepsilon^{m_1+1-i}H_i(t,\varepsilon s)\overset{(i)}{w_\varepsilon}(s)ds, \tag{9}$$

$$\overset{(n)}{w_\varepsilon}(\tau)+A_1(\varepsilon\tau)\overset{(n-1)}{w_\varepsilon}(\tau)+\varepsilon A_2(\varepsilon\tau)\overset{(n-2)}{w_\varepsilon}(\tau)+\ldots+\varepsilon^{n-1}A_n(\varepsilon\tau)w_\varepsilon(\tau)=0 \tag{10}$$

By the degree of ε formally expanding $H_i(t,\varepsilon s), i=\overline{0,m_1+1}$ into a Taylor series at the point $(t,0)$:

$$H_i(t,\varepsilon s)=H_i(t,0)+\varepsilon sH_i^{'}(t,0)+\frac{(\varepsilon s)^2}{2!}H_i^{''}(t,0)+\ldots+\frac{(\varepsilon s)^k}{k!}H_i^{(k)}(t,0)+\ldots\quad i=\overline{0,m_1+1} \tag{11}$$

Use (11) in (9), equating coefficients of like powers of ε , for the regular part $y_k(t), k=0,1,2,....$ we arrive the following equalities:

$$A_1y_0^{(n-1)}(t)+\sum_{k=2}^{n}A_k(t)y_0^{(n-k)}(t)=F(t)+\int_0^1\sum_{i=0}^{m_1+1}H_i(t,x)y_0^{(i)}(x)dx+\int_0^\infty H_{m_1+1}(t,0)\overset{(m_1+1)}{w_0}(s)ds$$

where $\int_0^\infty H_{m_1+1}(t,0)\overset{(m_1+1)}{w_0}(s)ds=-H_{m_1+1}(t,0)\overset{(m_1)}{w_0}(0)$

denote by

$$\Delta_0(t)=H_{m_1+1}(t,0)\Delta_0,\quad \Delta_0=-\overset{(m_1)}{w_0}(0). \tag{12$_0$}$$

for determining the coefficient $y_0(t)$, we obtain the integro-differential equation

$$A_1 y_0^{(n-1)}(t)+\sum_{k=2}^{n} A_k(t) y_0^{(n-k)}(t)=F(t)+\int_0^1 \sum_{i=0}^{m_1+1} H_i(t,x) y_0^{(i)}(x)dx+\Delta_0(t), \qquad (13_0)$$

where $\Delta_0(t)$ is defined by formula (12_0).

For determining the coefficients $y_k(t),\ k=1,2,....$ we obtain the integro-differential equation

$$A_1(t) y_k^{(n-1)}(t)+\sum_{i=2}^{n} A_i(t) y_k^{(n-i)}(t)=F_k(t)+\int_0^1 \sum_{i=0}^{m_1+1} H_i(t,x) y_k^{(i)}(x)dx+\Delta_k(t), \qquad (13_k)$$

where

$$\Delta_k(t)=H_{m_1+1}(t,0)\Delta_k,\ \ \Delta_k=-\overset{(m_1)}{w_k}(0) \qquad (12_k)$$

and $F_k(t)$ is known function, can be written as

$$F_k(t)=\int_0^\infty \sum_{j=1}^{k} \frac{s^j}{j!} H_{m_1+1}^{(j)}(t,0)\overset{(m_1+1)}{w_{k-j}}(s)ds+\int_0^\infty \sum_{i=1}^{k}\sum_{j=0}^{k-i} \frac{s^j}{j!} H_{m_1+1-i}^{(j)}(t,0)\overset{(m_1+1-i)}{w_{k-i-j}}(s)ds-y_{k-1}^{(n)}(t),$$
$$k=\overline{1,m_1+1} \qquad (14)$$
$$F_k(t)=\int_0^\infty \sum_{j=1}^{k} \frac{s^j}{j!} H_{m_1+1}^{(j)}(t,0)\overset{(m_1+1)}{w_{k-j}}(s)ds+\int_0^\infty \sum_{i=1}^{m_1+1}\sum_{j=0}^{k-i} \frac{s^j}{j!} H_{m_1+1-i}^{(j)}(t,0)\overset{(m_1+1-i)}{w_{k-i-j}}(s)ds-y_{k-1}^{(n)}(t),$$
$$k\geq m_1+2.$$

The values $\Delta_k(t),\Delta_k,k\geq 0$ are called respectively *the initial jumps of the integral terms and solutions.*

By the degree of ε formally expanding $A_i(\varepsilon\tau),\ i=\overline{1,n}$ into a Taylor series at the point 0:

$$A_i(\varepsilon\tau)=A_i(0)+\varepsilon\tau A_i'(0)+\frac{(\varepsilon\tau)^2}{2!}A_i''(0)+...+\frac{(\varepsilon\tau)^k}{k!}A_i^{(k)}(0)+...\ \ i=\overline{1,n}. \qquad (15)$$

Use (15) in (10), equating coefficients of like power of ε on both sides (10), we get the equations for the boundary layer functions $w_k(\tau),\quad k=0,1,2,\dots$

$$\overset{(n)}{w_0}(\tau)+A_1(0)\overset{(n-1)}{w_0}(\tau)=0, \qquad (16_0)$$
$$\overset{(n)}{w_k}(\tau)+A_1(0)\overset{(n-1)}{w_k}(\tau)=\Phi_k(\tau), \qquad (16_k)$$

where $\Phi_k(\tau)$ is known function, can be written as

$$\Phi_k(\tau)=\begin{cases} -\sum_{j=1}^{k}\frac{\tau^j}{j!}A_1^{(j)}(0)\overset{(n-1)}{w_{k-j}}(\tau)-\sum_{m_1=0}^{k-1}\sum_{j=0}^{m_1}\frac{\tau^j}{j!}A_{k+1-m_1}^{(j)}(0)\overset{(n-1+m_1-k)}{w_{m_1-j}}(\tau),\ k=\overline{1,n-1}, \\ \\ -\sum_{j=1}^{k}\frac{\tau^j}{j!}A_1^{(j)}(0)\overset{(n-1)}{w_{k-j}}(\tau)-\sum_{m_1=k+1-n}^{k-1}\sum_{j=0}^{m_1}\frac{\tau^j}{j!}A_{k+1-m_1}^{(j)}(0)\overset{(n-1+m_1-k)}{w_{m_1-j}}(\tau),\quad k\geq n \end{cases} \qquad (17)$$

To determine uniquely the terms $y_k(t)$ and $w_k(\tau)$ of the asymptotic, we use (5) in (6) and taking into account boundary condition (2)

$$\sum_{j=0}^{m_i}\alpha_{ij}[y_0^{(j)}(0)+\varepsilon y_1^{(j)}(0)+\ldots+\varepsilon^{m_1-j}(\overset{(j)}{w_0}(0)+\varepsilon \overset{(j)}{w_1}(0)+\ldots)]+ \tag{18}$$

$$+\sum_{j=0}^{l_i}\beta_{ij}\left[y_0^{(j)}(1)+\varepsilon y_1^{(j)}(1)+\ldots+\varepsilon^{m_1-j}\left(\overset{(j)}{w_0}\left(\frac{1}{\varepsilon}\right)+\varepsilon \overset{(j)}{w_1}\left(\frac{1}{\varepsilon}\right)+\ldots\right)\right]=a_i\ ,\ i=\overline{1,n}\ .$$

In (18) $\overset{(j)}{w_k}\left(\frac{1}{\varepsilon}\right), k=0,1,\ldots$ it is not taken into account, it cannot be compared than any degree of ε. Equating the coefficients at zero degrees of ε in (18) and in view of (12_0), we have

$$h_1 y_0(t)=a_1+\alpha_{1,m_1}\Delta_0,\ h_i y_0(t)=a_i,\quad i=\overline{2,n} \tag{19$_0$}$$

Thus, the main coefficient $y_0(t)$ of the regular part of the asymptotic and the initial jump of the solution Δ_0 are determined from the problem (13$_0$), (19$_0$).

For determining the coefficient $w_0(\tau)$, we have the initial condition $\Delta_0=-\overset{(m_1)}{w_0}(0)$ from (13$_0$), (19$_0$). Finding the missed initial condition for coefficient $w_0(\tau)$ we reduce the order of the equation (16$_0$) byintergrating from τ to ∞ and by virtue of the conditions $\overset{(i)}{w_0}(\infty)=0, i=\overline{0,n-1}$. As a result, after $n-1-m$ -th step, we obtain equation $\overset{(m_1+1)}{w_0}(0)=-A_1(0)\overset{(m_1)}{w_0}(0)$. From this equation as $\tau=0$,we determine the initial condition $\overset{(m+1)}{w_0}(0)=-A_1(0)\overset{(m)}{w_0}(0)$.Continuing this process lowering the degree of equation (16$_0$), we obtain the following initial conditions for $w_0(\tau)$:

$$\overset{(i)}{w_0}(0)=(-1)^{m_1+1-i}\frac{\Delta_0}{A_1(0))^{m_1-i}}\ ,\ \ i=\overline{0,n-1}. \tag{20$_0$}$$

Thus, the main coefficient $w_0(\tau)$ of the boundary layer part of the asymptotic is determined from the problem (16$_0$), (20$_0$).

Thus, the zeroth approximation of the asymptotic expansion is completely constructed. In the k -th approximation, for determining the boundary conditions of the coefficient $y_k(t)$, $k=1,2,\ldots$, we compare the coefficients of the same powers of the parameter ε. As a result, we obtain the following initial conditions for $y_k(t)$:

$$h_1 y_k=\begin{cases}\alpha_{1,m_1}\Delta_k-\sum\limits_{j=1}^{k}\alpha_{1,m_1-j}\overset{(m_1-j)}{w_{k-j}}(0),\ k=\overline{1,m_1}\ ,\\ \alpha_{1,m_1}\Delta_k-\sum\limits_{j=1}^{m_1}\alpha_{1,m_1-j}\overset{(m_1-j)}{w_{k-j}}(0),\ k\ge m_1+1\end{cases} \tag{19$_k$}$$

$$h_i y_k=\begin{cases}0,\ k=\overline{1,m_1-m_i-1},\\ -\sum\limits_{j=0}^{k-m_1+m_i}\alpha_{i,m_1-j}\overset{(m_i-j)}{w_{k-m_1+m_i-j}}(0),\quad k=\overline{m_1-m_i,m_1}\ ,\quad i=\overline{2,n}\\ -\sum\limits_{j=0}^{m_i}\alpha_{i,m_i-j}\overset{(m_i-j)}{w_{k-m_1+m_i-j}}(0),\quad k\ge m_1+1\end{cases}$$

From (13_k), (19_k) we determine $y_k(t), \Delta_k, k \geq 1$.

Now, we will be determine the initial conditions for the coefficient $w_k(\tau), k \geq 1$. In order to find the missing of the equation (16_k) by virtue of the conditions $\overset{(i)}{w_k}(\infty) = 0, i = \overline{0, n-1}$. Then, we get the initial conditions for determining $w_k(\tau), k \geq 1$:

$$\overset{(i)}{w_k}(0) = (-A_1(0))^{i-m_1} \overset{(m_1)}{w_k}(0) +$$

$$+(-1)^{n-1-i} \int_0^{\infty} \sum_{j=n-1-m_1}^{n-2-i} \frac{s^j}{j!} (A_1(0))^{j-(n-1-i)} \Phi_k(s) ds, \quad i = \overline{0, m_1},$$

$$\overset{(i)}{w_k}(0) = (-A_1(0))^{i-m_1} \overset{(m_1)}{w_k}(0) + \tag{20_k}$$

$$+(-1)^{n-i} \int_0^{\infty} \sum_{j=n-1-i}^{n-2-m_1} \frac{s^j}{j!} (A_1(0))^{j-(n-1-i)} \Phi_k(s) ds, \quad i \geq m_1 + 1$$

Thus, the k -th approximation of the asymptotic is completely constructed.

Results and discussion. *Theorem. Let functions* $A_i(t), F(t) \in C^{N+n-m_1}[a,b], i = \overline{1,n}$ *and conditions (C2) - (C7) hold. Then for sufficiently small* ε *the boundary value problem (1) and (2) has an unique solution on the* $0 \leq t \leq 1$ *and that is expressed by the formula*

$$y(t,\varepsilon) = \bar{y}_N(t,\varepsilon) + R_N(t,\varepsilon), \tag{21}$$

where $\bar{y}_N(t,\varepsilon)$ *is defined by the formula*

$$\bar{y}_N(t,\varepsilon) = \sum_{k=0}^{N} \varepsilon^k y_k(t) + \varepsilon^{m_1} \sum_{k=0}^{N+n-1-m_1} \varepsilon^k w_k(\tau), \quad \tau = \frac{t}{\varepsilon}, \tag{22}$$

and for the remainder term the estimates are valid

$$\left| R_N^{(i)}(t,\varepsilon) \right| \leq C\varepsilon^{N+1}, \quad i = \overline{0, n-1}, \quad 0 \leq t \leq 1. \tag{23}$$

where $C > 0$ *is a some constant independent of* ε *.*

Proof. We construct the N -th partial sum (22) of the expansion (5),(6).The function $\bar{y}_N(t,\varepsilon)$ satisfies problem (1), (2) with accuracy of order $O(\varepsilon^{N+1})$, i.e.

$$L_\varepsilon \bar{y}_N(t,\varepsilon) = F(t) + \int_0^1 \sum_{i=0}^{m_1+1} H_i(t,x) \bar{y}_N^{(i)}(x,\varepsilon) dx + O(\varepsilon^{N+1}),$$

$$h_i \bar{y}_N(t,\varepsilon) = a_i + O(\varepsilon^{N+1}), \quad i = 1, n \tag{24}$$

Denote by $y(t,\varepsilon) = \bar{y}_N(t,\varepsilon) + R_N(t,\varepsilon)$. Then for the remainder $R_N(t,\varepsilon)$ we obtain the problem as follows

$$\begin{cases} L_\varepsilon R_N(t,\varepsilon) = \int_0^1 \sum_{i=0}^{m_1+1} H_i(t,x) R_N^{(i)}(x,\varepsilon)dx + O(\varepsilon^{N+1}), \\ \\ h_i R_N(t,\varepsilon) = O(\varepsilon^{N+1}), \quad i = 1,n \end{cases} \tag{25}$$

We apply the asymptotic estimation of the solution of the problem (1),(2) to the problem (25). Then we obtain the estimates

$$\left| R_N^{(j)}(t,\varepsilon) \right| \le C\varepsilon^{N+1} + C\varepsilon^{N+1+m_1-j} \exp\left(-\gamma \frac{t}{\varepsilon}\right), \quad j = \overline{0, n-1} \quad 0 < t \le 1 \tag{26}$$

This means that estimates $R_N^{(m_1+1)}(t,\varepsilon) = O(\varepsilon^N),\ldots, R_N^{(n-1)}(t,\varepsilon) = O(\varepsilon^{N-n+2+m_1})$ is valid at point $t = 0$, i.e. The required estimates do not hold. To obtain the necessary estimates, we consider the equalities

$$y^{(m_1+1)}(t,\varepsilon) = y_N^{(m_1+1)}(t,\varepsilon) + R_N^{(m_1+1)}(t,\varepsilon), \quad y^{(m_1+1)}(t,\varepsilon) = y_{N+1}^{(m_1+1)}(t,\varepsilon) + R_{N+1}^{(m_1+1)}(t,\varepsilon) \tag{27}$$

Hence, equating the right-hand sides of (27), we get

$$R_N^{(m_1+1)}(t,\varepsilon) = y_{N+1}^{(m_1+1)}(t,\varepsilon) - y_N^{(m_1+1)}(t,\varepsilon) + R_{N+1}^{(m_1+1)}(t,\varepsilon), \tag{28}$$

where $\bar{y}_{N+1}^{(m_1+1)}(t,\varepsilon) - \bar{y}_N^{(m_1+1)}(t,\varepsilon) = \varepsilon^{N+1} y_{N+1}^{(m_1+1)}(t) + \varepsilon^{N+n-1-m_1} \overset{(m_1+1)}{w_{N+n-m_1}}(\tau)$ and the remainder term $R_{N+1}^{(m_1+1)}(t,\varepsilon)$ in (28) satisfies the estimate $\left| R_{N+1}^{(m_1+1)}(t,\varepsilon) \right| \le C\varepsilon^{N+2} + C\varepsilon^{N+1} \exp\left(-\gamma \frac{t}{\varepsilon}\right)$ Thus, we obtain the required estimates: $\left| R_N^{(m_1+1)}(t,\varepsilon) \right| \le C\varepsilon^{N+1}$. Similarly, considering the equalities

$$y^{(n-1)}(t,\varepsilon) = y_N^{(n-1)}(t,\varepsilon) + R_N^{(n-1)}(t,\varepsilon), \quad y^{(n-1)}(t,\varepsilon) = y_{N+n-1-m_1}^{(n-1)}(t,\varepsilon) + R_{N+n-1-m_1}^{(n-1)}(t,\varepsilon) \tag{29}$$

Hence, equating the right-hand sides of (29), we obtain

$$R_N^{(n-1)}(t,\varepsilon) = y_{N+n-1-m_1}^{(n-1)}(t,\varepsilon) - y_N^{(n-1)}(t,\varepsilon) + R_{N+n-1-m_1}^{(n-1)}(t,\varepsilon), \tag{30}$$

where

$$\bar{y}_{N+n-1-m_1}^{(n-1)}(t,\varepsilon) - \bar{y}_N^{(n-1)}(t,\varepsilon) = \varepsilon^{N+1} y_{N+1}^{(n-1)}(t) + \ldots + \varepsilon^{N+n-1-m_1} \overset{(n-1)}{y_{N+n-1-m_1}}(t) +$$

$$+ \varepsilon^{N+1} \overset{(n-1)}{w_{N+n-m_1}}(\tau) + \ldots + \varepsilon^{N+n-1-m_1} \overset{(n-1)}{w_{N+2(n-1-m_1)}}(\tau),$$

The remainder term $R_{N+n-1-m_1}^{(n-1)}(t,\varepsilon)$ in (30) satisfies the estimate $\left| R_{N+n-1-m_1}^{(n-1)}(t,\varepsilon) \right| \le C\varepsilon^{N+n-m_1} + C\varepsilon^{N+1} \exp\left(-\gamma \frac{t}{\varepsilon}\right)$. Thus, we obtain the required estimates $\left| R_N^{(n-1)}(t,\varepsilon) \right| \le C\varepsilon^{N+1}$. Theorem is proved.

Conclusion. We investigated asymptotic expansion of solution of general boundary value problem with initial jumps for higher-order singularly perturbed integro - differential equation with any degree of accuracy with respect to a small parameter have been constructed.

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СИНГУЛЯРЛЫ АУЫТҚЫҒАН ИНТЕГРАЛДЫ-ДИФФЕРЕНЦИАЛДЫҚ ТЕҢДЕУ ҮШІН БАСТАПҚЫ СЕКІРІСТІ ШЕТТІК ЕСЕБІ ШЕШІМІНІҢ АСИМПТОТИКАЛЫҚ ЖІКТЕЛУІ

Аннотация: қарастырылып отырған кесіндінің сол жақ шетінде $m_1 -$ ші ретті бастапқы секірісі бар $n-$ ші ретті сингулярлы ауытқыған Фредгольм сызықты интегралды-дифферециалдық теңдеулері үшін бөлінбеген шеттік есеп қарастырылады. Шешімнің асимтотикалық жіктелуінің регулярлы және шекаралық қабатты мүшелері анықталды. Асимптотиканың регулярлы мүшелері әдеттегі ауытқымаған теңдеулерден бөлек интегралдық мүшелердің бастапқы секірістері деп аталатын қосымша мүшелері бар интегралды-дифференциалдық теңдеулер түрінде құрылды. Осы бастапқы секірістердің шамалары анықталды. Асимптотиканың регулярлы мүшелерінің шеттік шарттарында $m_1 -$ ші ретті туындының бастапқы секірістері деп аталатын қосымша мүшелер болады. Сонымен, асимтотиканың регулярлы мүшелерін анықтау үшін сызықты интегралды-дифференциалдық теңдеулерге арналған қосымша параметрлі шеттік есептер алынды. Шешімнің асимптотикасының шекаралық қабатты мүшелерін анықтау үшін тұрақты коэффициентті біртекті және біртекті емес дифференциалдық теңдеулерге арналған бастапқы есептер алынды. Асимптотиканың шекаралық қабатты мүшелері үшін экспоненциалды бағалаулар алынды. Шешімнің бар болуы, даралығы және асимптотикалық түрде берілуіне теорема келтірілген. Асимптотиканың қалдық мүшесі бағаланды. Берілген сингулярлы ауытқыған интегралды-дифференциалдық теңдеулерге арналған шеттік есеп шешімінен құрылған асимптотикалық жуықтау барлық кесіндіде бірқалыпты сипатта болады.

Түйін сөздер: сингулярлы ауытқу, кіші параметр, асимптотикалық жіктелу, бастапқы секіріс, шекаралық қабат.

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АСИМПТОТИЧЕСКОЕ РАЗЛОЖЕНИЕ РЕШЕНИЙ КРАЕВЫХ ЗАДАЧ С НАЧАЛЬНЫМИ СКАЧКАМИ ДЛЯ СИНГУЛЯРНО ВОЗМУЩЕННЫХ ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ

Аннотация: рассматривается неразделенная краевая задача для сингулярно возмущенных линейных интегро - дифференциальных уравнений $n-$ гопорядка с интегральными членами Фредгольма, обладающая на левом конце рассматриваем огоотрезка явлением начального скачка $m_1 -$ гопорядка. Определены регулярные и погран слойные части асимптотического разложения решений. Регулярные члены асимптотики построены в видеинтегро – дифференциальных уравнений, отличающихся от обычных не возмущенных интегро-дифференциальных уравнений наличием дополнительных слагаемых, называемых начальными скачками интегральных членов. Определены величины этих начальных скачков. Краевые условия для регулярных членов асимптотики также содержат дополнительные слагаемые, называемые начальными скачками производных $m_1 -$ гопорядка. Тем самым, для определения регулярных членов асимптотики получаются крае выеза дачи для линейных интегро – дифференциальных уравнений с дополнительным параметром. Для определения погран слойных членов асимптотического разложения решений получены начальные задача для однородных и неоднородных обыкновенных дифференциальных уравнений с постоянными коэффициентами. Получены экспоненциальные оценки для погран слойных членов асимптотики. Сформулирована теорема существования, единственности и оба симптотическом представлении решений с оценкой остаточного члена асимптотики. Установлено, что построенное асимптотическое приближение к решению исходной сингулярно возмущенной интегро – дифференциальной краевой задачи носитравномерный характер на всем рассматриваемом отрезке.

Ключевые слова: сингулярное возмущение, малый параметр, асимптотическое разложение, начальный скачок, погранслой.

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ФИЗИКА

ИНФОРМАТИКА

МАТЕМАТИКА

СОДЕРЖАНИЕ

ФИЗИКА

ИНФОРМАТИКА

МАТЕМАТИКА

CONTENTS

PHYSICS

COMPUTER SCIENCE

MATHEMATICS

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